Seat No. ________________

HO-003-1162002

M. Sc. (Sem. II) Examination

April - 2023

Mathematics : CMT-2002

(Complex Analysis)

Faculty Code : 003
Subject Code : 1162002

Time : $2\frac{1}{2}$ / Total Marks : **70**

Instructions :
(1) All questions are compulsory.
(2) Each question carries equal marks.
(3) Figure on the right indicate allotted marks.

1 Answer any seven short questions. **7×2=14**

(1) Define : Stereographic projection, $T^{-1} : S \to \mathbb{C}_\infty$. What is its inverse?

(2) Define : Chordal metric on $\mathbb{C}_\infty$.

(3) Is the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \overline{z}$ differentiable? Justify it.

(4) Define : Fixed point. If $\lambda \in \mathbb{C}, \lambda \neq 0, \lambda \neq 1$ then find the fixed points of the bilinear transformation S defined by $S_z = \lambda z$.

(5) Find the bilinear transformation taking, $i \to 1, 0 \to \infty, -i \to 0$.

(6) Define with example : Branch of multi–valued function.

(7) State and prove, triangular inequality for the two complex numbers.

(8) Find the radius of converges of power series, $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

(9) If $G \subset \mathbb{C}$ is region, $f : G \to \mathbb{C}$ is continuous with primitive and $\gamma, \sigma : [a, b] \to G$ are rectifiable such that $\gamma(a) = \sigma(a)$, and $\gamma(b) = \sigma(b)$ then prove that, $\int_\gamma f = \int_\sigma f$.

(10) Prove that : If $f : G \to \mathbb{C}$ is differentiable then it is continuous.

2 Attempt any two. **2×7=14**

(a) State and prove : Fundamental theorem of calculus of line integral.

(b) State and prove : Maximum Modulus Theorem.

(c) State and prove : Cauchy Goursat's Theorem.

3 Attempt followings. **2×7=14**

(1) Prove : Cauchy's theorem for an open disc. (Let $a \in \mathbb{C}$, $R > 0, f : B(a, R) \to \mathbb{C}$ be analytic and γ be a closed rectifiable curve in $B(a, R)$. Then prove that, $\int_\gamma f = 0$).

(2) Derive, the formula of stereographic projection and Chordal metric on $\mathbb{C}_\infty$.

OR

3 Attempt followings. **2×7=14**

(1) Let $G \subset \mathbb{C}$ be open, $f : G \to \mathbb{C}$ be analytic, $a \in G$, $r > 0$, $\bar{B}(a, r) \subseteq G$ and $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(t) = a + r.e^{it}$, $\forall t \in [0, 2\pi]$. Then prove that,

$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z} dw, \forall z \in \mathbb{C}$ and

$|z - a| < r$, $\bar{B}(a, r) = \{z \in \mathbb{C} / |z - a| \leq r\}$.

(2) Let $f(z) = 1/\left(z - \frac{1}{2} - i\right)\left(z - 1 - \frac{3}{2} - i\right)\left(z - 1 - \frac{i}{2}\right)\left(z - \frac{3}{2} - i\right)$ and γ is the polygon [0, 2, 2 + 2*i*, 2*i*, 0]. Find, $\int_\gamma f$.

4 Attempt any two. **2×7=14**

(1) Show that, the set $M = \{S / S$ is a bilinear transformation$\}$ is a group under composition.

(2) Prove that, if $G \subset \mathbb{C}$ and $H \subset \mathbb{C}$ are open $f : G \to \mathbb{C}$ be continuous, $g : H \to \mathbb{C}$ be differentiable with $g'(x) \neq 0; \forall x \in H$ and $f(G) \subset H$, $g(f(z)) = z; \forall z \in G$ then f is differentiable and $f'(z) = \frac{1}{g'(f(z))}; \forall z \in G.$

(3) Prove that, for an analytic function $f : G \to \mathbb{C};$ where G be an open connected subset of $\mathbb{C}$ and $G^* = \{\bar{z} / z \in G\}$ then $f^* : G^* \to \mathbb{C}$ defined by $f^*(z) = \overline{f(\bar{z})}$, $\forall z \in G^*$ is analytic.

5 Attempt any two. **7×2=14**

(a) If γ is a rectifiable curve in $\mathbb{C}$, $f:\{\gamma\}\to\mathbb{C}$ is continuous and $c\in\mathbb{C}$ then prove that, $\int_{\gamma} f(z)dz = \int_{\gamma+c} f(z-c)\,dz.$

(b) Prove that, $\int_0^{2\pi} \frac{e^{is}}{e^{is}-z} ds = 2\pi,\ \forall z\in\mathbb{C}, |z|<1.$

(c) If $\gamma:[a, b]\to\mathbb{C}$ is a rectifiable path and $f:\{\gamma\}\to\mathbb{C}$ is continuous then prove that,

$$\left|\int_{\gamma} f\right| \le \int_{\gamma} |f||dz| \le V(\gamma)\bullet \sup_{z\in\{\gamma\}} |f(z)|.$$

(d) Evaluate : $\int_{\gamma} \frac{e^{z}-e^{-z}}{z^{n}} dz;$ where $n\in\mathbb{N}$ and $\gamma(t)=e^{it}, \forall t\in[0, 2\pi].$